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Generalized natural mechanical systems of two degrees of freedom with quadratic integrals

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Abstract. This paper is devoted to the problem of constructing integrable mechanical systems with two degrees of freedom whose Lagrangians contain terms linear in the velocities and whose second integrals have the form of polynomials of the second degree in the velocity variables in which the coefficients depend only on the coordinates. The solution of this problem is known only in the case of reversible systems, for which the linear terms do not affect Lagrange's equations of motion.

Using the method developed in a previous paper we classify all the possible irreversible systems into three types according to a certain normal form of the line element on the configuration space. The most general systems of the first two types are constructed. Several many-parameter systems of the third type are also found.

Some of the new cases are found to be generalizations, by the introduction of some additional parameters, to well known integrable problems in particle and rigid body dynamics. Mechanical interpretation is also given for some other cases.

1. Introduction

The mechanical system under consideration here is the system with two degrees of freedom characterized by a Lagrangian of the form

$$L=\tfrac{1}{2}\sum_{i,j=1}^{2} g_{ij}\dot{q}_i\dot{q}_j+\sum_{i=1}^{2} g_i\dot{q}_i-V \tag{1.1}$$

where g_{ij}, g_i and V are functions of q_1, q_2 only.

The Lagrangians which contain terms linear in the velocities are met in many different situations. Some examples follow.

(i) When the order of an arbitrary natural system with n degrees of freedom and $n-2$ cyclic coordinates is reduced according to Routh's procedure. An example is the rigid body fixed from one point with a symmetric rotor fixed in it by means of cylindrical hinges, if this sytem, called a gyrostat, moves under the action of forces which have an axis of symmetry that passes through the fixed point (see e.g. [1]). To this type also belong systems with intrinsic cyclic motions corresponding to an infinite number of degrees of freedom. A rigid body moving in a liquid [2] and (or) containing holes completely filled with liquid in vortex motion is an example [3]. The same applies for systems whose components are forced to perform cyclic or stationary motions (motors, forced motions of liquids in circuits, etc.). Forces that give rise to linear terms in (1.1) due to any of the above reasons are usually called 'gyroscopic' (e.g. [4]).

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(ii) When the original natural system has some electrically charged components which move under the action of a stationary magnetic field (Lorentz interaction). This is the case of forces 'with velocity-dependent potential' [5], 'with zero potential' [6] or 'with generalized potential' [7].

(iii) When the motion of the system is referred to rotating axes. That is, the case of 'inertial' or 'Coriolis' forces [4-7].

Depending on the origin of the linear terms in (1.1), the mechanical system under consideration is sometimes called 'reduced', 'non-natural', 'generalized conservative' or 'a system with velocity-dependent potential'. We shall call it a 'generalized-natural' system with no regard to its origin. This conforms with the well agreed definition of a natural system (e.g. [6-9]) when $g_1 = g_2 = 0$. It also leaves the general names above for the cases of more general dependence of the Lagrangian on the velocities.

Now, we shall consider some useful properties of our system.

(i) If $\partial g_1/\partial q_2 = \partial g_2/\partial q_1$, then (1.1) can be written as

$$L = L_0 + \mathrm{d}/\mathrm{d}t\, f(q_1, q_2)$$

where L_0 has no linear terms. The equations of motion derived from the Lagrangians L and L_0 are the same. They are also invariant under time reversal. This means that if $q(t)$ is a solution then $q(-t)$ is also a solution. In that case, we call our system 'reversible'. Otherwise, we call it 'irreversible' (see e.g. [9]). It is obvious that these terms should not be confused with the thermodynamic irreversibility.

(ii) According to a theorem of Birkhoff [10], one can always find isothermal coordinates x, y (say) in which the Lagrangian (1.1) takes the form

$$L = \tfrac{1}{2}\Lambda(\dot{x}^2 + \dot{y}^2) + l_1\dot{x} + l_2\dot{y} - V. \tag{1.2}$$

The new Lagrangian involves four functions of the position instead of six in (1.1). introducing a new independent variable τ by the relation

$$\mathrm{d}t = \Lambda\, \mathrm{d}\tau \tag{1.3}$$

we can replace (1.2) on any fixed level of its Jacobi integral† by

$$L = \tfrac{1}{2}(\mathring{x}^2 + \mathring{y}^2) + l_1\mathring{x} + l_2\mathring{y} + U \tag{1.4}$$

where

$$U = \Lambda(h - V) \tag{1.5}$$

h is Jacobi's constant and the circle denotes differentiation with respect to τ. Then, as was noted by Birkhoff, the equations of motion will have the form

$$\overset{\circ\circ}{x} + \Omega\mathring{y} = \frac{\partial U}{\partial x} \qquad \overset{\circ\circ}{y} - \Omega\mathring{x} = \frac{\partial U}{\partial y} \tag{1.6a}$$

$$\mathring{x}^2 + \mathring{y}^2 - 2U = 0 \tag{1.6b}$$

where

$$\Omega = \frac{\partial l_1}{\partial y} - \frac{\partial l_2}{\partial x}. \tag{1.7}$$

† The Hamiltonian of the system expressed in Lagrangian variables. It coincides with the total energy only for reversible systems.

Note that in (1.6*b*) a superfluous Jacobi constant of (1.6*a*) is set equal to zero, while the Jacobi constant h for the original system (1.2) enters as a parameter in the new potential $-U$. Note also that the system is reversible if and only if $\Omega = 0$.

Equations (1.6) contain only two functions Ω and U and hence they furnish the simplest basis for the search of additional integrals.

(iii) The last form (1.6) of the equations of motion is invariant under conformal mappings of the complex plane $z = x + iy$ [6, 8-10]. In fact, a change of the variables

$$z = z(\zeta) \qquad \zeta = \xi + i\eta \qquad d\tau = \left|\frac{dz}{d\zeta}\right|^2 d\tilde{\tau} \tag{1.8}$$

transforms the system (1.6) to the form

$$\xi'' + \tilde{\Omega}\eta' = \frac{\partial \tilde{U}}{\partial \xi} \qquad \eta'' - \tilde{\Omega}\xi' = \frac{\partial \tilde{U}}{\partial \eta} \tag{1.9a}$$

$$\xi'^2 + \eta'^2 - 2\tilde{U} = 0 \tag{1.9b}$$

where

$$\tilde{\Omega} = \left|\frac{dz}{d\zeta}\right|^2 \Omega \qquad \tilde{U} = \tilde{\Lambda}(h - V) \qquad \tilde{\Lambda} = \left|\frac{dz}{d\zeta}\right|^2 \Lambda$$

and primes denote differentiation with respect to $\tilde{\tau}$. That is one more advantage of equations (1.6) over equations of the Lagrange or Hamilton types.

The object of the present paper is to construct integrable generalized natural mechanical systems which admit a quadratic additional integral. This problem has a long history, mainly concerned with the reversible case $\Omega \equiv 0$ (see e.g. [11]). Our main interest will therefore be directed towards the irreversible case.

Birkhoff raised and completely solved the problem of finding all possible pairs U, Ω for which the system (1.6) admits an additional integral linear in the velocities [10]. He has also found the general expression for the function U which allows the existence of a quadratic integral in the reversible case.

The special case when $\Lambda = \text{constant}$ and $\Omega \equiv 0$ characterizes the problem of motion of a particle under the action of potential forces in the Euclidean plane. It was treated much earlier. Bertrand [12] reduced it to a single partial differential equation which was solved later by Darboux [13] (see also Whittaker [7]). Degenerate cases overlooked by Darboux were noted by several recent authors (see e.g. Hietarinta [11]).

In our work [1] a method was developed for the construction of mechanical systems for which a conditional first integral exists in the form of a polynomial of arbitrary degree in the velocities. This method, which generalizes that of Birkhoff, has proved effective in constructing several, in part new, integrable problems, in the dynamics of particles and rigid bodies, for which the degree of the additional integral ranges up to the fourth [1, 14].

A similar method was developed independently by Hall [15] for the case of plane motion of a particle ($\Lambda \equiv 1$). In view of the complexity of Hall's equations, his method was used only in the reversible case ($\Omega \equiv 0$). In our method, additional transformations of the type (1.8) are used. This leads to significant simplification in the form of the governing equations and of their solutions.

In section 2 below, we derive the equations which determine all the possible mechanical systems of the type under consideration which admit a quadratic second integral. However, we consider further only the cases when that integral exists on all the levels of Jacobi's constant. These cases admit a simple classification into three types according to the structure of the function $\tilde{\Lambda}$ which determines the metric of the configuration space. These three types are investigated in sections 4, 5 and 6, respectively. A brief note concerning the reversible case is given in section 3. Finally, we devote a separate section to the discussion of some problems in rigid body dynamics. We reduce their equations of motion to the normal form (1.9) and explore their connection to the results of the former sections.

2. Formulation of the problem

Now we consider the equations of motion of our mechanical system in the form (1.6). Let these equations admit an additional integral in the form

$$\Im = A\mathring{x}^2 + B\mathring{x}\mathring{y} + C\mathring{y}^2 + D\mathring{x} + E\mathring{y} + F \tag{2.1}$$

where the coefficients are functions of x and y only and do not depend on the parameter h. The form of the integral (2.1) can be simplified as follows.

(i) Using (1.6b) we eliminate $\mathring{y}^2$ to get

$$\Im = A_1\mathring{x}^2 + B\mathring{x}\mathring{y} + D\mathring{x} + E\mathring{y} + F_1 \tag{2.2}$$

where $A_1 = A - C$, $F_1 = F + 2CU$. Note that F_1 depends on h linearly.

(ii) Noting that

$$(a\mathring{x} + b\mathring{y})^2 = (a\mathring{x} + b\sqrt{2U - \mathring{x}^2})^2$$
$$= (a^2 - b^2)\mathring{x}^2 + 2ab\mathring{x}\mathring{y} + 2b^2U$$

we can always reduce the integral to the form

$$\Im = (a\mathring{x} + b\mathring{y})^2 + D\mathring{x} + E\mathring{y} + F_2 \tag{2.3}$$

where $(a + \mathrm{i}b)^2 = A_1 + \mathrm{i}B$, and F_2 is also linear in h.

(iii) differentiating (2.3) with respect to τ and using the equations of motion (1.6) and equating to zero the coefficients of the highest powers of $\mathring{x}$ and $\mathring{y}$ we get

$$\frac{\partial a}{\partial x} - \frac{\partial b}{\partial y} = 0 \qquad \frac{\partial a}{\partial y} + \frac{\partial b}{\partial x} = 0$$

i.e. $a + \mathrm{i}b$ is an analytic function in a certain domain of the plane of the complex variable z. We now define the new coordinates ξ and η and the new independent variable $\tilde{\tau}$ as in (1.8) such that

$$\zeta = \int \frac{\mathrm{d}z}{a + \mathrm{i}b} \qquad \mathrm{d}\tau = |a + \mathrm{i}b|^2\,\mathrm{d}\tilde{\tau}. \tag{2.4}$$

This transforms the equations of motion to the form (1.9) and the integral (2.3) to the final form

$$\Im = \tfrac{1}{2}\xi'^2 + P\xi' + Q\eta' + R. \tag{2.5}$$

It suffices, therefore, to consider systems of the type (1.9) which admit an integral of the form (2.5) where P, Q and R are functions of ξ, η; P, Q are independent of h while R depends on h linearly.

It is easy to deduce the set of conditions for (2.5) to be a first integral of (1.9) in the form of four equations which must be satisfied by the four functions P, Q, $\tilde{U}$ and $\tilde{\Omega}$:

$$\begin{aligned} &P_\xi - Q_\eta = 0 \qquad P_\eta + Q_\xi = \tilde{\Omega} \\ &P\tilde{U}_\xi + Q\tilde{U}_\eta + 2Q_\eta \tilde{U} = 0 \qquad (\tilde{\Omega}P)_\xi + (\tilde{\Omega}Q)_\eta + \tilde{U}_{\xi\eta} = 0 \end{aligned} \tag{2.6}$$

while for R we get the quadrature

$$R = \int \tilde{\Omega} P \, \mathrm{d}\eta - (\tilde{\Omega} Q + \tilde{U}_\xi)\, \mathrm{d}\xi. \tag{2.7}$$

The general solution of (2.6) determines all the possible mechanical systems whose motion is described by equations of the type (1.9) and for which an additional integral of the form (2.5) exists. Generally speaking, this integral is conditional, i.e. it is valid only on a single level of Jacobi's constant which can be taken as the zero level. These systems can be generalized by the introduction of the transformation (2.4) followed by a general point transformation to generate all the possible cases of the original system (1.1) which admit a quadratic integral of the most general possible form.

On the other hand, the transformation (2.4) changes the Lagrangian (1.4) to

$$L = \tfrac{1}{2}(\xi'^2 + \eta'^2) + \tilde{l}_1 \xi' + \tilde{l}_2 \eta' + \tilde{U} \tag{2.8}$$

where $\tilde{l}_1$, $\tilde{l}_2$ satisfy

$$\tilde{l}_{1\eta} - \tilde{l}_{2\xi} = \tilde{\Omega}$$

and the Jacobi constant of the resulting equations is set equal to zero. Comparing (2.8) with the expression for $\tilde{\Omega}$ in (2.6) we find that a possible Lagrangian is

$$\tilde{L} = \tfrac{1}{2}(\xi'^2 + \eta'^2) + P\xi' - Q\eta' + \tilde{U}. \tag{2.9}$$

All other possible Lagrangians differ from this only by an expression $\mathrm{d}f/\mathrm{d}\tilde{\tau}$, f is an arbitrary function of ξ and η.

The system (2.6) can be reduced further by the substitution

$$P = \phi_\eta \qquad Q = \phi_\xi \qquad \tilde{\Omega} = \phi_{\xi\xi} + \phi_{\eta\eta} \tag{2.10}$$

to a pair of equations

$$\phi_\eta \tilde{U}_\xi + \phi_\xi \tilde{U}_\eta + 2\tilde{U}\phi_{\xi\eta} = 0 \tag{2.11a}$$

$$\left[\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right](\phi_\xi \phi_\eta) + \tilde{U}_{\xi\eta} = 0 \tag{2.11b}$$

in the two functions ϕ and $\tilde{U}$.

We have not yet been able to find the general solution of (2.11) and hence to determine all systems that admit a quadratic integral at least on a single level of Jacobi's constant. However, a considerable simplification occurs in the most interesting case when the integral (2.5) exists on every level of Jacobi's constant. Noting that the functions P, Q and hence ϕ, do not depend on h, and substituting for $\tilde{U}$ in (2.11) the expression $\tilde{\Lambda}\,(h - V)$ and equating to zero the coefficients of different powers of h, we arrive at the following system:

$$\tilde{\Lambda}_{\xi\eta}=0 \tag{2.12a}$$

$$\phi_\eta\tilde{\Lambda}_\xi+\phi_\xi\tilde{\Lambda}_\eta+2\tilde{\Lambda}\phi_{\xi\eta}=0 \tag{2.12b}$$

$$\phi_\eta V_\xi+\phi_\xi V_\eta=0 \tag{2.12c}$$

$$\left[\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right](\phi_\xi\phi_\eta)-(\tilde{\Lambda}V)_{\xi\eta}=0. \tag{2.12d}$$

It can be shown that if the integral (2.5) exists just for two different values of h then (2.12) are satisfied and the same integral is valid for all other values of h.

The general solution of (2.12a) is

$$\tilde{\Lambda}=\varepsilon[\lambda(\xi)-\mu(\eta)] \tag{2.13}$$

where λ, μ are arbitrary functions and ε is an arbitrary constant introduced for future convenience. This means that a mechanical system can admit a quadratic integral only if the metric on its configuration space admits reduction in some generalized coordinates to a Liouville metric, including two possible degenerate cases when one of the two functions λ, μ is a constant or both of them are constants. The integral takes its simplest form (2.5) just in these coordinates. It is natural that the solution of the remaining equations (2.12b–d) depends mainly on which of these three types the solution (2.13) belongs to. For the irreversible case the three possible types will be treated in sections 4–6. We now begin by considering the reversible case.

3. The reversible case

When $\Omega\equiv 0$ we have to set $\phi=0$ and hence (2.12) gives

$$V=\frac{\phi_1(\xi)-\phi_2(\eta)}{\lambda(\xi)-\mu(\eta)} \tag{3.1}$$

where ϕ_1,ϕ_2 are arbitrary functions. In the variables ξ, η, $\tilde{\tau}$ the mechanical system under consideration has the separable Lagrangian

$$\tilde{L}=\tfrac{1}{2}[\xi'^2+\eta'^2]+h[\lambda-\mu]-\phi_1+\phi_2 \tag{3.2}$$

restricted to its zero level of the total energy. This is equivalent to the unconditioned Lagrangian

$$L=\tfrac{1}{2}(\lambda-\mu)[\dot{\xi}^2+\dot{\eta}^2]-\frac{\phi_1-\phi_2}{\lambda-\mu}$$

of the Liouville type.

This result is applied to rigid body dynamics in [1].

3.1. *The case of an Euclidean plane*

If in (1.2) we set $\Lambda=1$, the original configuration space is the ordinary plane. In the transformed system we have

$$\tilde{\Lambda}=\left|\frac{dz}{d\zeta}\right|^2=\lambda(\xi)-\mu(\eta).$$

This means that the function $z(\zeta)$ must satisfy

$$\frac{\partial^2}{\partial\xi\partial\eta}\left(\left|\frac{dz}{d\zeta}\right|^2\right)\equiv i\left[\frac{\partial^2}{\partial\zeta^2}-\frac{\partial^2}{\partial\bar{\zeta}^2}\right]\left[\frac{dz}{d\zeta}\frac{d\bar{z}}{d\bar{\zeta}}\right]=0$$

which is equivalent to

$$\frac{1}{dz/d\zeta}\frac{d^3z}{d\zeta^3}=\alpha$$

α being an arbitrary real constant. The general solution of the last equation is

$$\zeta=\int\frac{dz}{(\alpha z^2+\beta z+\gamma)^{1/2}} \tag{3.3}$$

where β and γ are new arbitrary constants which can always be made real by a suitable change of the axes.

Now, there is no difficulty in recognizing that ξ, η as defined in (3.3) are in general the elliptic coordinates in the xy-plane with foci at the two branching points of the integrand. Thus we arrive in a simple and unified manner at Darboux' result (erroneously attributed to Whittaker in many recent works) and its degenerate cases: $\alpha=0$, $\beta^2-4\alpha\gamma=0$ and $\alpha=\beta=0$ when (3.3) defines parabolic, polar and Cartesian coordinates, respectively. These cases were overlooked by Darboux. They have been noted much later in [16].

4. Irreversible systems. The configuration space is an Euclidean plane referred to Cartesian coordinates

We begin by considering the simplest case $\Lambda=$ constant or, without loss of generality, $\Lambda=1$ and $\zeta=z$. It is not hard then to obtain the general solution of equations (2.12) as

$$\begin{aligned}\phi&=q(x)+p(y)\\ V&=-\tfrac{1}{2}[\alpha(q-p)^3+(\beta_1+\beta_2)(q-p)^2]-h_1(q-p)\end{aligned} \tag{4.1}$$

where q and p are determined from the relations

$$\begin{aligned}x&=\pm\int\frac{dq}{(\alpha q^3+\beta_1q^2+\gamma_1q+\delta_1)^{1/2}}\\ y&=\pm\int\frac{dp}{(-\alpha p^3+\beta_2p^2+\gamma_2p+\delta_2)^{1/2}}\end{aligned} \tag{4.2}$$

α, β_1, β_2, γ_1, γ_2, δ_1, δ_2 and h_1 are arbitrary constants.

Hence we obtain the expressions

$$\begin{aligned}&P=p'(y)\qquad Q=q'(x)\\ &\Omega=\tfrac{1}{2}[3\alpha(q^2-p^2)+2(\beta_1q+\beta_2p)+\gamma_1+\gamma_2].\end{aligned} \tag{4.3}$$

This result may be interpreted as a case of motion of a particle in the Euclidean xy-plane. In that context, it was noted earlier [17]. A new interpretation is given in section 6 below.

The functions $p(y)$, $q(x)$ as determined from (4.2) are in the general case elliptic with two independent sets of invariants. Apart from certain degenerate cases these functions are periodic and hence V, Ω are periodic functions in both x and y directions. Each of $q(x)$ and $p(y)$ has at most two real branches but at most one of these branches can be bounded. The last kind of branches is the most significant since a real pole of q or p leads to the appearance of a singular line for V and Ω in the xy-plane.

The bounded case occurs when $Q^2(q)$ and $P^2(p)$ have only real roots. Let us denote these roots by q_1, q_2, q_3, p_1, p_2 and p_3, where $p_1 > p_2 > p_3$ and $q_1 > q_2 > q_3$. For determinacy, let α also be positive. The bounded branch of $q(p)$ lies in the interval $[q_3, q_2]([p_2, p_1])$.

They have the explicit expressions

$$q = q_3 + (q_2 - q_3) sn^2(\alpha_1 x, k_1) \tag{4.4a}$$

$$p = p_1 - (p_1 - p_2) sn^2(\alpha_2 y, k_2) \tag{4.4b}$$

where

$$\alpha_1^2 = \alpha(q_1 - q_3)/4 \qquad k_1^2 = (q_2 - q_3)/(q_1 - q_3)$$

$$\alpha_2^2 = \alpha(p_1 - p_3)/4 \qquad k_2^2 = (p_1 - p_2)/(p_1 - p_3).$$

5. Irreversible systems: the metric has the structure of a metric on a surface of revolution

Let us now choose for (2.12a) a solution of the form

$$\tilde{\Lambda} = \varepsilon \mu(\eta). \tag{5.1}$$

This makes the metric formally similar to that on a surface of revolution, but it does not indicate, however, the possibility of realizing the configuration space as a surface of revolution embedded in Euclidean 3D space. With this choice the solution of equation (2.12b) can be written as

$$\phi = \frac{g(\xi)}{\sqrt{\mu}} - \int \frac{v'(\mu)}{\mu} \, d\mu \tag{5.2}$$

where g and v are functions to be determined. Substituting into (2.12c) we obtain

$$V = V(\psi) \qquad \psi = \sqrt{\mu}\, g + v. \tag{5.3}$$

It remains to determine the four functions $\mu(\eta)$, $g(\xi)$, $v(\mu)$ and $V(\psi)$ from the last equation (2.12d). Substituting into this equation we find that V must have the form

$$V = \frac{1}{\varepsilon}(b\psi - \tfrac{1}{2} c_3 \psi^2) \tag{5.4}$$

g is a solution of

$$g''(\xi) + c_0 g(\xi) = 0 \tag{5.5}$$

μ is determined from

$$\eta = \pm \frac{1}{2} \int \frac{d\mu}{\mu(c_3\mu^3 + c_2\mu^2 + c_1\mu + c_0)^{1/2}} \tag{5.6}$$

and v satisfies the equation

$$8(c_3\mu^3+c_2\mu^2+c_1\mu+c_0)v'''(\mu)+12(3c_3\mu^2+2c_2\mu+c_1)v''(\mu)$$
$$+6(3c_3\mu+c_2)v'(\mu)-3c_3v(\mu)+3b=0 \tag{5.7}$$

where b, c_0, c_1, c_2 and c_3 are arbitrary constants.

Equation (5.5) can readily be solved in terms of elementary functions. The whole problem therefore reduces to solving the linear ordinary differential equation (5.7). This depends totally on the coefficients c_0, c_1, c_2, c_3 of the polynomial under the square root in (5.6). Let us denote by μ_1, μ_2 and μ_3 the roots of the same polynomial. We get the following cases.

5.1. The case $c_3 \neq 0$

In this case, depending on the relative values of the three roots, the solution of (5.7) can be written as:

(i) for $\mu_1 \neq \mu_2 \neq \mu_3$,

$$v=\frac{b}{c_3}+c_4\sqrt{\mu-\mu_1}+c_5\sqrt{\mu-\mu_2}+c_6\sqrt{\mu-\mu_3} \tag{5.8a}$$

where each of the arbitrary constants c_4, c_5 and c_6 is taken to be real or imaginary so that the corresponding term in (5.8a) is real. If the polynomial has only one real root (μ_3 say), then the roots μ_1 and μ_2 are complex conjugate and so we must take the arbitrary constants c_4, c_5.

(ii) for $\mu_1=\mu_2 \neq \mu_3$,

$$v=\frac{b}{c_3}+c_4\sqrt{\mu-\mu_1}+\frac{c_5}{\sqrt{\mu-\mu_1}}+c_6\sqrt{\mu-\mu_3} \tag{5.8b}$$

(iii) for $\mu_1=\mu_2=\mu_3$,

$$v=\frac{b}{c_3}+c_4\sqrt{\mu-\mu_1}+\frac{c_5}{\sqrt{\mu-\mu_1}}+\frac{c_6}{(\mu-\mu_1)^{3/2}} \tag{5.8c}$$

5.2. The case $c_3=0$, $c_2 \neq 0$

The polynomial has only two roots μ_1, μ_2 and the solution of (5.7) is

(i) $$v=-\frac{b\mu}{2c_2}+c_4+c_5\sqrt{\mu-\mu_1}+c_6\sqrt{\mu-\mu_2} \qquad \mu_2 \neq \mu_1. \tag{5.9a}$$

The constant c_4 is real while c_5, c_6 are taken as above so that v is real.

(ii) $$v=-\frac{b\mu}{2c_2}+c_4+c_5\sqrt{\mu-\mu_1}+\frac{c_6}{\sqrt{\mu-\mu_1}} \qquad \mu_2=\mu_1. \tag{5.9b}$$

5.3. The case $c_3=c_2=0$, $c_1 \neq 0$

$$v=-\frac{b\mu^2}{8c_1}+c_4+c_5\mu+c_6\sqrt{c_1\mu+c_0}. \tag{5.10}$$

5.4. The case $c_3=c_2=c_1=0$, $c_0\neq 0$

In this case

$$v=-\frac{b\mu^3}{16c_0}+c_4+c_5\mu+c_6\mu^2. \tag{5.11}$$

The solution of the problem of the present section is now complete. However, we have not yet found physical or mechanical interpretation for all the cases obtained. Some special versions can be interpreted as problems of motion of a symmetric rigid body or of a particle on a smooth surface of revolution. We give here an interpretation for the case that corresponds to the choice (5.11).

First we note that for general values of the parameters $c_0,\ldots,c_3$, the metric characterized by the relation (5.6), namely

$$\mathrm{d}s^2=\varepsilon\mu(\mathrm{d}\xi^2+\mathrm{d}\eta^2) \tag{5.12}$$

has the Gaussian curvature

$$\begin{aligned}\kappa&=-\frac{1}{2\varepsilon\mu}(\ln\mu)_{\eta\eta}\\ &=-\frac{1}{\varepsilon}(3c_3\mu^2+2c_2\mu+c_1).\end{aligned} \tag{5.13}$$

It is obvious that the case $(c_3=c_2=c_1=0)$ is the only one when the configuration space of our mechanical system can be interpreted as an Euclidean plane. The coordinates ξ, η are related to polar coordinates in that plane. In fact, performing transformation to the z-plane

$$z=\mathrm{e}^{-\mathrm{i}\sqrt{c_0}\zeta/4} \tag{5.14}$$

and using the above formulae we arrive at the problem of motion of a particle described by the equations

$$\ddot{x}+\Omega\dot{y}=-V_x \qquad \ddot{y}-\Omega\dot{x}=-V_y \tag{5.15}$$

where

$$V=Ax+By-(cr^2+abr^4+a^2r^6) \tag{5.16a}$$

$$\Omega=2b+6ar^2 \tag{5.16b}$$

A, B, a, b and c are arbitrary constants and $r=|z|$. The additional quadratic integral for this problem can be easily constructed with the use of the formulae given above and effecting the transformation (5.14). We write down this integral in the following final form which can be checked directly:

$$\begin{aligned}\Im=&(x\dot{y}-y\dot{x}-br^2-\tfrac{3}{2}ar^4)[a(x\dot{y}-y\dot{x}-br^2-\tfrac{3}{2}ar^4)-c]\\ &+\tfrac{1}{2}(A\dot{y}-B\dot{x})-(b+ar^2)(Ax+By).\end{aligned} \tag{5.17}$$

The potential V can be considered as due to a combination of uniform and central fields. The function Ω can be understood as the intensity of a magnetic field, assuming the particle to have an electric charge. The constant part of Ω also arises as a result of uniform rotation of the plane about its normal. The integral (5.17) degenerates into a linear one when $a=0$.

6. Irreversible systems. The metric is a non-degenerate Liouville one

Now we take the solution of (2.12*a*) in the general form (2.13) with $\lambda'(\xi)\ \mu'(\eta)\neq 0$. In this case we can use λ, μ as variables. Let

$$\lambda'^2(\xi)=F(\lambda) \qquad \mu'^2(\eta)=G(\mu) \tag{6.1a}$$

so that

$$\xi=\pm\int\frac{\mathrm{d}\lambda}{\sqrt{F(\lambda)}} \qquad \xi=\pm\int\frac{\mathrm{d}\mu}{\sqrt{G(\mu)}}. \tag{6.1b}$$

Equations (2.12*b*) and (2.12*c*) can be written as

$$\phi_\mu-\phi_\lambda+2(\lambda-\mu)\phi_{\lambda\mu}=0 \tag{6.2a}$$

$$\phi_\mu V_\lambda+\phi_\lambda V_\mu=0. \tag{6.2b}$$

The equation of the characteristics of (6.2*b*) is

$$\phi_\lambda\,\mathrm{d}\lambda-\phi_\mu\,\mathrm{d}\mu=0. \tag{6.3}$$

It is easy to verify, by virtue of (6.2*a*), that (6.3) has the integrating factor $(\lambda-\mu)$. Hence the potential V can be expressed in the form $V=V(\psi)$, where ψ is connected to ϕ by the relations

$$\psi_\lambda=(\lambda-\mu)\phi_\lambda \qquad \psi_\mu=-(\lambda-\mu)\phi_\mu \tag{6.4}$$

and hence satisfies the equation

$$\psi_\lambda-\psi_\mu+2(\lambda-\mu)\psi_{\lambda\mu}=0. \tag{6.5}$$

The function ψ will be used below instead of ϕ. In particular (2.10) is replaced by

$$\begin{aligned}
&P=-\sqrt{G(\mu)}\,\frac{\psi_\mu}{\lambda-\mu} \qquad Q=\sqrt{F(\lambda)}\,\frac{\psi_\lambda}{\lambda-\mu}\\
&\Omega=F(\lambda)\frac{\partial}{\partial\lambda}\left(\frac{\psi_\lambda}{\lambda-\mu}\right)-G(\mu)\frac{\partial}{\partial\mu}\left(\frac{\psi_\mu}{\lambda-\mu}\right)+\frac{[F'(\lambda)\psi_\lambda-G'(\mu)\psi_\mu]}{2(\lambda-\mu)}.
\end{aligned} \tag{6.6}$$

Now we provide an interesting interpretation of the functions ϕ and ψ. In fact if we introduce new variables $\rho=\lambda-\mu$, $Z=\mathrm{i}(\lambda+\mu)$ then (6.2*a*), (6.5) and (6.4) reduce to

$$\phi_{\rho\rho}+\frac{1}{\rho}\,\phi_\rho+\phi_{ZZ}=0 \tag{6.7a}$$

$$\psi_{\rho\rho}-\frac{1}{\rho}\,\psi_\rho+\psi_{ZZ}=0 \tag{6.7b}$$

$$(-\mathrm{i}\psi)_Z=-\rho\phi_\rho \qquad (-\mathrm{i}\psi)_\rho=\rho\phi_Z. \tag{6.7c}$$

Moreover, we can imagine Z to be the axis of cylindrical coordinates in some (complex) 3D space and ρ as the radial distance of the current point from that axis. Equations (6.7) are exactly those satisfied by the velocity potential ϕ and Stokes' stream function $(-\mathrm{i}\psi)$ of a vritual flow of an ideal incompressible fluid, symmetric around the Z-axis (see e.g. [18]).

This formal analogy makes it possible to write down some simple solutions of (6.6) inspired by known axisymmetric hydrodynamic flows.

We note that, in addition to equation (6.5) for ψ, only one equation remains to be satisfied by the four functions $\psi(\lambda,\mu)$, $F(\lambda)$, $G(\mu)$ and $V(\psi)$. That is (2.12d) which now takes the form

$$[F''(\lambda)+G''(\mu)]\frac{\psi_\lambda\psi_\mu}{\lambda-\mu}+3\left\{\left[\frac{\psi_\lambda\psi_\mu}{(\lambda-\mu)^2}\right]_\lambda F'(\lambda)+\left[\frac{\psi_\lambda\psi_\mu}{(\lambda-\mu)^2}\right]_\mu G'(\mu)\right\}$$
$$+2\left\{\left[\frac{\psi_\lambda\psi_\mu}{(\lambda-\mu)^2}\right]_{\lambda\lambda} F(\lambda)+\left[\frac{\psi_\lambda\psi_\mu}{(\lambda-\mu)^2}\right]_{\mu\mu} G(\mu)\right\}$$
$$+2\varepsilon(\lambda-\mu)[V''(\psi)\psi_\lambda\psi_\mu+3V'(\psi)\psi_{\lambda\mu}]=0. \tag{6.8}$$

6.1. *The analogue of a uniform flow*

We begin with the simplest hydrodynamic problem, namely, we shall consider a uniform flow in the Z-direction. This corresponds to the choice (compare e.g. [18])

$$\phi=J(\lambda+\mu) \qquad \psi=\tfrac{1}{2}J(\lambda-\mu)^2 \tag{6.9}$$

where J is a constant. It can be easily seen that this choice satisfies (6.8) if and only if

$$V=-\frac{J^2}{2}[\alpha(\lambda-\mu)^2+(\beta_1+\beta_2)(\lambda-\mu)]-\frac{h_1}{\lambda-\mu}$$
$$F(\lambda)=\alpha\lambda^3+\beta_1\lambda^2+\gamma_1\lambda+\delta_1 \tag{6.10}$$
$$G(\mu)=-\alpha\mu^3+\beta_2\mu^2+\gamma_2\mu+\delta_2$$

where α, β_1, β_2, γ_1, γ_2, δ_1, δ_2 and h_1 are arbitrary constants. This also gives

$$\Omega=\frac{J}{2}[3\alpha(\lambda^2-\mu^2)+2(\beta_1\lambda+\beta_2\mu)+\gamma_1+\gamma_2]. \tag{6.11}$$

The expressions (6.10) and (6.11) characterize a mechanical system which can be obtained from that of section 4 by a change of time $dt\to dt/(\lambda-\mu)$ and interchanging h, h_1.

As in the previous sections we shall try to obtain a mechanical interpretation of some special cases of this result. First, it will be useful to calculate the Gaussian curvature of the configuration space whose metric is $ds^2=\varepsilon(\lambda-\mu)(d\xi^2+d\eta^2)$. We find

$$\kappa=\frac{1}{4\varepsilon}\left[\alpha+\frac{2(\beta_1+\beta_2)\lambda\mu+(\gamma_1+\gamma_2)(\lambda+\mu)+2(\delta_1+\delta_2)}{(\lambda-\mu)^3}\right]. \tag{6.12}$$

So we recognize the following simple cases:

(i) Let $\beta_1+\beta_2=\gamma_1+\gamma_2=\delta_1+\delta_2=0$, $\alpha=4$, $\varepsilon=1$ (say) and let the roots of the polynomial $F(\lambda)$ all be real and distinct. Then $\kappa=1$ and the configuration space can be realized as a sphere of unit radius. The variables λ, μ are the coordinates of confocal cones on that sphere. So we have an integrable case of motion of a particle on the sphere under the action of certain potential and gyroscopic forces. However, a more interesting interpretation belongs to rigid body dynamics. The equivalence of the two problems was studied in [19].

(ii) Let $\alpha=\beta_1+\beta_2=\gamma_1+\gamma_2=\delta_1+\delta_2=0$, and let $F(\lambda)$ have two simple real roots. In that case $\kappa=0$. The configuration space can be identified (but not uniquely) with the Euclidean plane. Going back to Cartesian coordinates x, y we find that for this mechanical system the equations of motion can be written in the form (5.15) with

$$V=\frac{a}{r_1 r_2} \qquad \Omega=\text{constant} \tag{6.13a}$$

where a is a constant and r_1, r_2 are the distances between the particle and two fixed points $(\pm c, 0)$. This system admits the integral

$$\begin{aligned}\Im=&[(x-c)\dot{y}-y\dot{x}][(x+c)\dot{y}-y\dot{x}]+\Omega[c^2(y\dot{x}+x\dot{y})+(x^2+y^2)(y\dot{x}-x\dot{y})]\\&-a\frac{c^2-x^2-y^2}{r_1r_2}+\frac{\Omega^2}{4}[(x^2+y^2)^2+2c^2(y^2-x^2)].\end{aligned} \tag{6.13b}$$

6.2. *Some cases with algebraic ψ*

The rest of the present section will be devoted to the construction of some classes of integrable mechanical systems for which the function ψ is symmetric in the arguments λ, μ that depend in a homogeneous manner on certain arbitrary parameters. The functions $F(\lambda)$ and $G(\mu)$ are rational. The results given below were found by trial, inspired partly by the structure of some known integrable problems in rigid body dynamics. It was not always easy to investigate equation (6.8) even for relatively simple choices for ψ, F, G and V. Thus, we do not claim, as in the previous sections, that we give here all the possible integrable systems of the type under consideration.

For brevity, each integrable system will be characterized by the corresponding functions ψ, F, G, V and Ω. This will be sufficient to construct the original Lagrangian of the system as well as its quadratic integral with the aid of the formulae in section 2 and at the beginning of the present section.

6.2.1. The generating case. The first of our systems is characterized by the choice of a solution of (6.5) in the form

$$\psi=2J\sqrt{(\lambda-\alpha)(\mu-\alpha)} \tag{6.14}$$

where α is a real constant. Substituting in (6.8) we obtain the equation

$$2\varepsilon J^2\left[V''(\psi)+3\frac{V'(\psi)}{\psi}\right]=H \tag{6.15}$$

where

$$\begin{aligned}H=&\frac{J^2}{(\lambda-\mu)^5}\{(\lambda-\mu)^2[F''(\lambda)+G''(\mu)]+6(\lambda-\mu)[F'(\lambda)+G'(\mu)]\\&+12[F(\lambda)+G(\mu)]\}.\end{aligned} \tag{6.16}$$

It is obvious that H must not have a singularity at $\lambda=\mu$. The necessary condition for that is

$$G(\mu)=-F(\mu). \tag{6.17}$$

On the other hand, H is a function only of ψ, so that

$$\frac{\partial(H,\psi)}{\partial(\lambda,\mu)}=0. \tag{6.18}$$

To solve this equation we first operate on both its sides by

$$\left(\frac{\partial^2}{\partial\lambda\partial\mu}\right)^3[\sqrt{(\lambda-\alpha)(\mu-\alpha)}(\lambda-\mu)^6].$$

We get the much simpler equation

$$(\mu-\alpha)F^{VI}(\mu)+F^{V}(\mu)=(\lambda-\alpha)F^{VI}(\lambda)+F^{V}(\lambda) \tag{6.19}$$

whose solution is

$$F(\lambda)=a_5\lambda^5+a_4\lambda^4+a_3\lambda^3+a_2\lambda^2+a_0\lambda+a_0+\frac{b}{\lambda-\alpha}. \tag{6.20}$$

It is easy to verify that this solution also satisfies (6.18) and hence it gives the general solution of that equation.

Substituting (6.20) and (6.17) into (6.15) and taking (6.14) into account we obtain a differential equation in V whose solution is

$$V=-\frac{a_5\psi^2}{8}+\frac{bJ^6}{8\psi^4}+\frac{c}{\psi^2} \tag{6.21}$$

where c is a new integration constant. Also from (6.6) we get

$$\Omega=\tfrac{1}{2}J(\lambda-\mu)\sqrt{(\lambda-\alpha)(\mu-\alpha)}\times\left[a_5[2(\lambda+\mu)+\alpha]+a_4-\frac{a_5\alpha^5+a_4\alpha^4+a_3\alpha^3+a_2\alpha^2+a_1\alpha+a_0}{(\lambda-\alpha)^2(\mu-\alpha)^2}-\frac{2b(\lambda+\mu-2\alpha)}{(\lambda-\alpha)^3(\mu-\alpha)^3}\right]. \tag{6.22}$$

A possible Lagrangian for the above system can be written as

$$L=\tfrac{1}{2}\varepsilon(\lambda-\mu)\left[\frac{\dot\lambda^2}{F(\lambda)}-\frac{\dot\mu^2}{F(\mu)}\right]-J\frac{\sqrt{-F(\lambda)F(\mu)}}{(\lambda-\mu)\sqrt{(\lambda-\alpha)(\mu-\alpha)}}\left\{\frac{(\lambda-\alpha)}{F(\lambda)}\dot\lambda-\frac{(\mu-\alpha)}{F(\mu)}\dot\mu\right\}-\frac{bJ^2}{2\varepsilon(\lambda-\alpha)^2(\mu-\alpha)^2}+\frac{a_5J^2}{2\varepsilon}(\lambda-\alpha)(\mu-\alpha)-\frac{c}{(\lambda-\alpha)(\mu-\alpha)}. \tag{6.23}$$

Special versions of this result are given interpretation in particle dynamics and in rigid body dynamics in section 7.

6.2.2. A case of a superposition of terms. The previous integrable case involved 10 arbitrary real parameters. It is interesting to note some variations of this system which involve the same number of 10 parameters. In all of them we have

$$F(\lambda)=a_5\lambda^5+\ldots+a_0 \qquad G(\mu)=-F(\mu) \qquad V=-a_5\psi^2/8\varepsilon. \tag{6.24}$$

The most general one of these systems is characterized by the choice of ψ as a linear combination of five terms of the type (6.14). That is

$$\psi=2\sum_{j=1}^{5}J_i\sqrt{(\lambda-\alpha_j)(\mu-\alpha_j)} \tag{6.25a}$$

provided that α_j ($j=1,\ldots,5$) are the roots of the fifth degree polynomial $F(\lambda)$, and J_j are arbitrary constants which may be chosen real or complex (conjugate in pairs) in such a way that ψ is real. The above form of ψ leads to the following expression for Ω:

$$\Omega = \tfrac{1}{2}(\lambda-\mu)\sum_{j=1}^{5} J_j\sqrt{(\lambda-\alpha_j)(\mu-\alpha_j)}[2a_5(\lambda+\mu)+a_5\alpha_j+a_4]. \tag{6.25b}$$

6.2.3. Some limiting cases. From the last case we can obtain several limiting cases by coalescing together the values of the different parameters α_j in various ways. For example, if two of the roots are equal, say $\alpha_4=\alpha_5$, then the last term in (6.25*a*) should be replaced by its derivative with respect to α_4. Thus we obtain the integrable system identified by

$$\psi = 2\sum_{j=1}^{4} J_j\sqrt{(\lambda-\alpha_j)(\mu-\alpha_j)}+J_5\frac{\lambda+\mu-2\alpha_4}{\sqrt{(\lambda-\alpha_4)(\mu-\alpha_4)}}. \tag{6.26a}$$

For this system we have

$$V = -\frac{a_5}{8\varepsilon}\psi^2 \tag{6.26b}$$

$$\Omega = \tfrac{1}{2}(\lambda-\mu)\left\{\sum_{j=1}^{4} J_j\sqrt{(\lambda-\alpha_j)(\mu-\alpha_j)}[2a_5(\lambda+\mu)+a_5\alpha_j+a_4]+\frac{J_5}{\sqrt{(\lambda-\alpha_4)(\mu-\alpha_4)}}\right.$$
$$\left.\times[2a_5(\lambda^2+\mu^2+\lambda\mu)+(a_4-\alpha_4a_5)(\lambda+\mu)-2\alpha_4(a_4+2\alpha_4a_5)]\right\}. \tag{6.26c}$$

In the sense of the hydrodynamic problem, the last term in (6.26*a*) (multiplied by $-i$) is just the Stokes' stream function for the flow due to a source at the point $\rho=0$, $Z=2i\alpha_4$. Each of the other four terms is Stokes' function for a more complicated flow due to a half-line source and a half-line sink separated by the point $2i\alpha_j$ on the Z-axis. We can still obtain more limiting cases by coalescing more roots to α_4 again or to other roots.

A root α repeated $(j+1)$ times leads to the replacement of a number of j terms in (6.25*a*) by terms of the form

$$\sum_{k=1}^{j} J_h\frac{\partial^k}{\partial\alpha^k}\sqrt{(\lambda-\alpha)(\mu-\alpha)}.$$

For example, let

$$F(\lambda)=a_5(\lambda-\alpha)^3(\lambda-a)(\lambda-b).$$

This means that, in addition to the two types of terms that appeared in (6.26*a*), the function ψ will contain a term like

$$\frac{J_4(\lambda-\mu)^2}{(\lambda-\alpha)^{3/2}(\mu-\alpha)^{3/2}}$$

and this results in a term

$$\frac{J_4a_5(\lambda-\mu)}{4(\lambda-\alpha)^{3/2}(\mu-\alpha)^{3/2}}[2(\lambda^3+\lambda^2\mu+\lambda\mu^2+\mu^3)-(a+b)(\lambda-\mu)^2$$
$$-2\alpha(\lambda+\mu)^2+12\alpha^2(\lambda+\mu)-8\alpha^2]$$

in the expression (6.26*b*) for Ω.

We have not yet found a concrete mechanical interpretation for any of the above integrable cases. The first step in that interpretation is a geometrical one. That is to identify a configuration space with the Liouville metric

$$ds^2=\varepsilon(\lambda-\mu)(d\xi^2+d\eta^2)=\varepsilon(\lambda-\mu)\left[\frac{d\lambda^2}{F(\lambda)}-\frac{d\mu^2}{F(\mu)}\right]. \tag{6.27}$$

After that we dress this space with the potential and gyroscopic forces characterized by the two functions V and Ω. For the system of section 6.2.2 (and also section 6.2.1 if $b=0$), F is a polynomial of the fifth degree and hence ξ and η are hyperelliptic integrals in λ, μ, respectively. When F has two equal roots, as in (6.26), these integrals become elliptic of the third kind. To this type belongs the metric on the reduced configuration space of the problem of motion of a rigid body about a fixed point under the action of axially symmetric forces (see e.g. [1, 4(c)]). As we shall see in section 7, the case characterized by (6.26) can be identified, after some restrictions on its parameters, as a very well known integrable case in rigid body dynamics.

6.2.4. A transformation of the above cases. Here we note a striking property of the basic system of the present section, namely the system described by the Lagrangian (6.23). For the sake of clarity we set $\alpha=0$, so that we consider further the system with the Lagrangian

$$L=\tfrac{1}{2}\varepsilon(\lambda-\mu)\left[\frac{\dot\lambda^2}{F(\lambda)}-\frac{\dot\mu^2}{F(\mu)}\right]-\frac{J\sqrt{-F(\lambda)F(\mu)}}{(\lambda-\mu)\sqrt{\lambda\mu}}\left[\frac{\lambda\dot\lambda}{F(\lambda)}-\frac{\mu\dot\mu}{F(\mu)}\right]$$
$$-\frac{bJ^2}{2\varepsilon\lambda^2\mu^2}+\frac{a_5J^2}{2\varepsilon}\lambda\mu-\frac{c}{\lambda\mu} \tag{6.28}$$

where

$$f(\lambda)=a_5\lambda^5+a_4\lambda^4+a_3\lambda^3+a_2\lambda^2+a_0\lambda+a_0+\frac{b}{\lambda}.$$

Transforming to new variables u, v, T according to the relations

$$\lambda=\frac{1}{u}\qquad \mu=\frac{1}{v}\qquad dt=\left(\frac{-1}{uv}\right)dT. \tag{6.29}$$

On a fixed level h of Jacobi's constant of the system, we reduce the Lagrangian (6.28) to

$$\Im=\tfrac{1}{2}\varepsilon(u-v)\left[\frac{\overset{*}{u}{}^2}{F(u)}-\frac{\overset{*}{v}{}^2}{F(v)}\right]-\frac{J\sqrt{-F(u)F(v)}}{(u-v)\sqrt{uv}}\left[\frac{u\overset{*}{u}}{F(u)}-\frac{v\overset{*}{v}}{F(v)}\right]$$
$$-\frac{a_5J^2}{2\varepsilon u^2v^2}+\frac{bJ^2}{2\varepsilon}uv+\frac{h}{uv} \tag{6.30}$$

where

$$f(u)=bu^5+a_0u^4+a_1u^3+a_2u^2+a_3u+a_4+\frac{a_5}{u}$$

and the asterisk denotes differentiation with respect to T. Thus, the structure of the Lagrangian (6.28) is invariant under the change of variables (6.29). All that happened is that the ordered set of parameters $\{a_5,\ldots,a_0,b\}$ is reversed and the two arbitrary constants c and h are interchanged. Jacobi's constant h for the old system enters as a parameter in the potential and the parameter c becomes Jacobi's constant for the new system.

The two problems characterized by the Lagrangians (6.28) and (6.30) are completely equivalent from the mathematical point of view. A special version of one of them always transforms to a special version of the other. However, the two corresponding problems can be physically quite different. As we shall see in section 7, we can use this result to establish equivalence between a problem of motion of a rigid body and a problem of motion of a particle on a smooth ellipsoid. The same transformation can also be applied to other cases of the present section.

6.3. Some cases of polynomial ψ

In the present subsection we introduce certain integrable systems for which all the functions ψ, F, V, and Ω have the simple form of polynomials in λ and μ.

(i) Let us take a solution of (6.6) in the form

$$\psi = J(\lambda+\mu) \qquad J = \text{constant}. \tag{6.31}$$

Substituting into (6.8) we get

$$V''(\psi) = -\frac{1}{2\varepsilon}\left\{\frac{F''(\lambda)+F''(\mu)}{(\lambda-\mu)^3} - 6\frac{F'(\lambda)-F'(\mu)}{(\lambda-\mu)^4} + 12\frac{F(\lambda)-F(\mu)}{(\lambda-\mu)^5}\right\}. \tag{6.32}$$

The left-hand side of the last equation is a function only of $(\lambda+\mu)$. Thus from the right-hand side we obtain

$$\begin{aligned}[F'''(\lambda)-F'''(\mu)](\lambda-\mu)^3 - 12[F''(\lambda)-F''(\mu)](\lambda-\mu)^2 \\ +60[F'(\lambda)-F'(\mu)](\lambda-\mu) - 120[F(\lambda)-F(\mu)] = 0.\end{aligned} \tag{6.33a}$$

Operating on both sides of this equation by $(\partial^2/\partial\lambda\partial\mu)^3$, we obtain the simple equation

$$F^{VI}(\lambda) - F^{VI}(\mu) = 0 \tag{6.33b}$$

whose solution is

$$F(\lambda) = a_6\lambda^6 + \ldots + a_1\lambda + a_0 \tag{6.34}$$

$a_6, \ldots, a_0$ are arbitrary constants. It can be verified that (6.34) also satisfies (6.33a) and hence it is its only solution. To complete the geometric side of the problem we write down the Gaussian curvature of the configuration space determined by (6.30) and (6.34):

$$\kappa = -\frac{1}{4\varepsilon}[2a_6(\lambda+\mu)(2\lambda^2+\lambda\mu+2\mu^2) + a_5(3\lambda^2+4\lambda\mu+3\mu^2) + 2a_4(\lambda+\mu) + a_3]. \tag{6.35}$$

Now, substituting (6.34) into (6.32) and integrating we get

$$V = -\tfrac{1}{2}J^2[a_6(\lambda+\mu)^3 + a_5(\lambda+\mu)^2] + N(\lambda+\mu) \tag{6.36a}$$

where N is a new integration constant. Also from (6.31) and (6.6)

$$\begin{aligned}\Omega = \tfrac{1}{2}J(\lambda-\mu)[2a_6(\lambda+\mu)(2\lambda^2+\lambda\mu+2\mu^2) \\ + a_5(3\lambda^2+4\lambda\mu+3\mu^2) + 2a_4(\lambda+\mu) + a_3].\end{aligned} \tag{6.36b}$$

Thus, choosing ψ in the form (6.31) we have constructed an integrable system that depends on nine parameters. It is interesting to note that for the system under consideration

$$\Omega = -2\varepsilon J(\lambda-\mu)\kappa \tag{6.37}$$

so that Ω vanishes together with the Gaussian curvature of the configuration space when $a_6 = a_5 = a_4 = a_3 = 0$. In that case the system degenerates into a Liouville one. If $a_3 \neq 0$, $a_j = 0$ for $j > 3$, and if (6.34) has three distinct real roots, then λ, μ can be interpreted as elliptic coordinates on the sphere. This result admits an interpretation as an integrable problem in the dynamics of a rigid body of complete dynamical symmetry in a liquid.

The function (6.31) is the simplest polynomial solution of (6.6). It represents the Stokes' function of the flow due to an infinite uniform line-source. The following variations of this result in which ψ is also a low-degree polynomial were obtained by trial. For each case we give ψ, V, $\tilde{\Omega}$ and the conditions on the coefficients in (6.34).

(ii)
$$a_6 = a_5 = a_4 = 0 \tag{6.38a}$$
$$\psi = J(\lambda + \mu) + J_1(\lambda - \mu)^2 \tag{6.38b}$$
$$V = -2a_3 J_1 \psi \tag{6.38c}$$
$$\tilde{\Omega} = \tfrac{1}{2}(\lambda - \mu)[6J_1 a_3(\lambda + \mu) + J(a_3 + 4a_2)]. \tag{6.38d}$$

(iii)
$$a_6 = a_5 = a_4 = a_3 = 0 \tag{6.39a}$$
$$\psi = J(\lambda + \mu) + J_1(\lambda - \mu)^2 + J_2(\lambda + \mu)(\lambda - \mu)^2 \tag{6.39b}$$
$$V = -4a_2 J_2 \psi \tag{6.39c}$$
$$\tilde{\Omega} = 2(\lambda - \mu)[3J_2 a_2(\lambda + \mu) + 2a_1 J_2 + a_2 J_1]. \tag{6.39d}$$

(iv)
$$a_6 = a_5 = a_4 = a_3 = a_2 = 0 \tag{6.40a}$$
$$\psi = J(\lambda + \mu) + (\lambda - \mu)^2[J_1 + J_2(\lambda + \mu)] \tag{6.40b}$$
$$V = -8a_1 J_3 \psi \tag{6.40c}$$
$$\tilde{\Omega} = 4(\lambda - \mu)[3J_3 a_1(\lambda + \mu) + 2a_0 J_3 + a_1 J_2]. \tag{6.40d}$$

Each of the last three systems involve six free parameters. For the last two systems (iii) and (iv) $\kappa = 0$. They can be interpreted as cases of motion of a particle in the Euclidean plane referred to elliptic and parabolic coordinates, respectively. In both cases, the equations of motion in the original xy-plane have the form (5.15). We now give the functions V and Ω together with the form of the additional integral in the original Cartesian variables.

In case (iii) we obtain the system for which

$$\Omega = 8[(J_1 + J_0)C^2 + 3r^2 J_1] \tag{6.41a}$$

$$V = -16J_1\{[3J_1 + 2J_0)r^2 - 4(J_1 + J_0)x^2 + Jr^2]C^4 + J_1 r^6$$
$$+ [(3J_1 + J_0)r^2 - 4J_1 x^2]C^2 r^2 + (J + J_1 + J_0)C^6\}. \tag{6.41b}$$

This system has the integral

$$+ a_5(3\lambda^2 + 4\lambda\mu + 3\mu^2) + 2a_4(\lambda + \mu) + a_3]. \tag{6.36b}$$

Thus, choosing ψ in the form (6.31) we have constructed an integrable system that depends on nine parameters. It is interesting to note that for the system under consideration

$$\Omega = -2\varepsilon J(\lambda - \mu)\kappa \tag{6.37}$$

$$\begin{aligned}R_1 = 4\{&4C^6[(3J_1+J_0)r^2-2x^2J_1]J-2(5J_1^2+6J_1J_0+2J_0^2)x^2\\&+(3J_1+2J_0)(3J_1+J_0)r^2]+2C^4[(27J_1^2+18J_1J_0+2J_0^2)r^4\\&-8(5J_1+3J_0)r^2x^2J_1+3Jr^4J_1+8x^4J_1^2]\\&+4[3(3J_1+J_0)r^2-10x^2J_1]C^2r^4J_1+9r^8J_1^2\}.\end{aligned}$$

Similarly, in case (iv) we obtain the system for which

$$\Omega = 2(6c_4x-c_3) \tag{6.42a}$$

$$V = 2c_4[2c_1x+2r^2(c_2+c_3x-2c_4x^2)-c_4r^4] \tag{6.42b}$$

$$\Im = (x\dot{y}-y\dot{x})\dot{y}+P_2\dot{x}+Q_2\dot{y}+R_2 \tag{6.42c}$$

where

$$P_2 = -2y(c_2+c_3x-3c_4x^2-c_4y^2)$$

$$Q_2 = c_1+2c_2x+c_3(3x^2+y^2)-2c_4x(5x^2+3y^2)$$

$$\begin{aligned}R_2 = 2[&c_1x(c_3-2c_4x)+r^2[c_2c_3-c_1c_4+x(c_3^2-4c_2c_4)\\&+2c_4x^2(4c_4x-3c_3)]+c_4r^4(4c_4x-c_3)].\end{aligned}$$

7. Application to rigid body dynamics

7.1. Reduction to the problem of plane motion of a particle

The most general and well studied problem in the dynamics of rigid bodies is that of the motion of a rigid body bounded by a multiconnected surface in an infinite medium of ideal incompressible fluid. This problem contains as special cases the problems of motion of a rigid body (or a gyrostat) about a fixed point under the action of uniform or approximate Newtonian field of attraction. In its general form the problem is described by the set of Lamb's equations [2] which are not derived from a Lagrangian function. A slightly modified form used by Kharlamov and others (see e.g. [20]) exhibit the same disadvantage. The same problem was investigated in detail in [4], where it was transformed into a special version of the problem of motion of an electrically charged gyrostat about a fixed point under the action of a superposition of Newtonian, Coulomb and Lorentz' forces which have a common axis of symmetry that passes through the fixed point. The last problem has the advantage of admitting equations of motion in the Lagrangian form, which are suitable for applying the method of the present work. We now proceed to write down the equations of motion of this problem and to make the necessary reduction to a system of two degrees of freedom.

Let $\boldsymbol{I} = \mathrm{diag}(A, B, C)$ be the inertia matrix of the body at the fixed point, $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ a unit vector in the direction of the axis of symmetry of the fields and $\boldsymbol{\omega}$ the angular velocity of the body, all being referred to the system of principal axes of $\boldsymbol{I}$, fixed in the body. Let also Ψ be the angle of precession measured around the axis in the direction of $\boldsymbol{\gamma}$ through the fixed point.

The Lagrangian of the system under consideration can be written as

$$L = \tfrac{1}{2}\boldsymbol{\omega I}\cdot\boldsymbol{\omega}+\boldsymbol{m}\cdot\boldsymbol{\omega}-V_0 \tag{7.1}$$

where V_0 is a scalar potential and $\boldsymbol{m}$ is a vector function which characterizes gyroscopic, inertial and Lorentz' forces. Both functions are independent of the angle Ψ.

As in [4(*b*)] the equations of motion of the system can be written in the Euler–Poisson form as

$$\dot{\boldsymbol{\omega}}\boldsymbol{I}+\boldsymbol{\omega}\times(\boldsymbol{\omega}\boldsymbol{I}+\boldsymbol{M})=\boldsymbol{\gamma}\times\frac{\partial V_0}{\partial\boldsymbol{\gamma}} \tag{7.2a}$$

$$\dot{\boldsymbol{\gamma}}+\boldsymbol{\omega}\times\boldsymbol{\gamma}=0 \tag{7.2b}$$

where

$$\boldsymbol{M}=\frac{\partial}{\partial\boldsymbol{\gamma}}(\boldsymbol{m}\cdot\boldsymbol{\gamma})-\left(\frac{\partial}{\partial\boldsymbol{\gamma}}\cdot\boldsymbol{m}\right)\boldsymbol{\gamma}. \tag{7.2c}$$

As configuration variables in the Lagrangian (7.1) we shall use the redundant variables γ_1, γ_2, γ_3, Ψ, subject to the obvious constraint

$$\gamma_1^2+\gamma_2^2+\gamma_3^2=1. \tag{7.3}$$

The angular velocity can be expressed in the form

$$\boldsymbol{\omega}=\dot{\Psi}\boldsymbol{\gamma}+\boldsymbol{N} \tag{7.4}$$

$$\boldsymbol{N}=\frac{(-\gamma_2\dot{\gamma}_3,\ \gamma_1\dot{\gamma}_3,\ \gamma_2\dot{\gamma}_1-\gamma_1\dot{\gamma}_2)}{\gamma_1^2+\gamma_2^2}.$$

The angle Ψ is a cyclic variable. The corresponding integral is

$$(\boldsymbol{\omega}\boldsymbol{I}+\boldsymbol{m})\cdot\boldsymbol{\gamma}=\text{constant}=f. \tag{7.5}$$

From (7.4) and (7.5) we get

$$\dot{\Psi}=\frac{1}{D}[f-\boldsymbol{m}\cdot\boldsymbol{\gamma}-\boldsymbol{\gamma}\boldsymbol{I}\cdot\boldsymbol{N}] \tag{7.6}$$

where

$$D=\boldsymbol{\gamma}\boldsymbol{I}\cdot\boldsymbol{\gamma}=A\gamma_1^2+B\gamma_2^2+C\gamma_3^2.$$

Ignoring Ψ in (7.1) we construct the Routhian

$$R=\frac{ABC}{2D}\left[\frac{\dot{\gamma}_1^2}{A}+\frac{\dot{\gamma}_2^2}{B}+\frac{\dot{\gamma}_3^2}{C}\right]+\boldsymbol{n}\cdot\dot{\boldsymbol{\gamma}}-V \tag{7.7a}$$

where

$$\left.\begin{aligned}V&=V_0+\frac{(f-\boldsymbol{m}\cdot\boldsymbol{\gamma})^2}{2D}\\ \boldsymbol{n}&=\frac{1}{D}\left[\boldsymbol{\gamma}\boldsymbol{I}\times\boldsymbol{m}+f\frac{\partial}{\partial\dot{\boldsymbol{\gamma}}}(\boldsymbol{\gamma}\boldsymbol{I}\cdot\boldsymbol{N})\right].\end{aligned}\right\} \tag{7.7b}$$

Now we introduce generalized coordinates on the sphere (7.3) that reduce the quadratic part of R in (7.7) to the Liouville form and hence reduce the equations of motion to the form (1.9). The transformation to these variables depends on the relative order of the parameters A, B and C. The case when these parameters are all different is discussed below. The case when only two of the moments are equal was found to be of no particular interest for our purpose here, since all known integrable problems with quadratic integrals in rigid body dynamics are in this case special versions of others that are valid for $A>B>C$.

7.2. The case of triaxial ellipsoid of inertia $A \neq B \neq C$

In this case the substitution

$$\gamma_1 = \left(\frac{BC(A-\lambda)(A-\mu)}{(A-B)(A-C)\lambda\mu}\right)^{1/2} \qquad \gamma_2 = \left(\frac{AC(\lambda-B)(B-\mu)}{(A-B)(B-C)\lambda\mu}\right)^{1/2}$$
$$\gamma_3 = \left(\frac{AB(\lambda-C)(\mu-C)}{(B-C)(A-C)\lambda\mu}\right)^{1/2} \qquad A \geqslant \lambda \geqslant B \geqslant \mu \geqslant C \tag{7.8}$$

together with the change of the independent variable

$$\mathrm{d}t = \tfrac{1}{4}(\lambda-\mu)\,\mathrm{d}\tau \tag{7.9}$$

reduce the equations of motion to the form

$$\xi'' + \Omega\eta' = \frac{\partial U}{\partial \xi} \qquad \eta'' - \Omega\xi' = \frac{\partial U}{\partial \eta} \tag{7.10a}$$

$$\xi'^2 + \eta'^2 - 2U = 0 \tag{7.10b}$$

where

$$\xi = \sqrt{ABC}\int \frac{\mathrm{d}\lambda}{\lambda\sqrt{(A-\lambda)(\lambda-B)(\lambda-C)}} \qquad \eta = \sqrt{ABC}\int \frac{\mathrm{d}\mu}{\mu\sqrt{(A-\mu)(B-\mu)(\mu-C)}} \tag{7.11}$$

$$U = \frac{\lambda-\mu}{4}(h-V) \tag{7.12}$$

$$\Omega = \frac{(\lambda-\mu)\sqrt{\lambda\mu}}{4ABC}\,[f[A+B+C-2(\lambda+\mu)]+S] \tag{7.13}$$

$$S = D\frac{\partial}{\partial \boldsymbol{\gamma}}\cdot\left[\frac{1}{D}\,\boldsymbol{\gamma}\times(\boldsymbol{\gamma I}\times\boldsymbol{m})\right]. \tag{7.14}$$

The following obvious identities will be useful in simplifying the expression (7.13) in concrete cases and in identifying integrable cases:

$$D = \frac{ABC}{\lambda\mu} \qquad \left(\frac{1}{B}+\frac{1}{C}\right)\gamma_1^2 + \left(\frac{1}{C}+\frac{1}{A}\right)\gamma_2^2 + \left(\frac{1}{A}+\frac{1}{B}\right)\gamma_3^2 = \frac{\lambda+\mu}{\lambda\mu}$$
$$|\boldsymbol{\gamma I}|^2 = D[A+B+C-(\lambda+\mu)]. \tag{7.15}$$

To identify integrable problems in rigid body dynamics among the cases of section 6 we have to look for systems for which, in accordance with (7.11),

$$F(\lambda) = \frac{1}{ABC}\,\lambda^2(A-\lambda)(\lambda-B)(\lambda-C). \tag{7.16}$$

There are three cases that satisfy this requirement of which two coincide with two well known general integrable cases of the motion of an asymmetric body in a liquid.

7.2.1. Reconstruction of Clebsch's case. The first case is that of section 6.2.1 if we set

$$\alpha = b = a_0 = a_1 = 0 \qquad a_2 = 1 \qquad a_3 = -\left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)$$

$$a_4 = \frac{A+B+C}{ABC} \qquad a_5 = \frac{-1}{ABC} \qquad \varepsilon = \tfrac{1}{4}.$$

In this case the functions V, Ω are given by

$$V = \frac{c}{\lambda\mu} + \frac{ABCJ^2}{8}\lambda\mu \tag{7.17a}$$

$$\Omega = \frac{J(\lambda-\mu)\sqrt{\lambda\mu}}{2ABC}[A+B+C-2(\lambda+\mu)]. \tag{7.17b}$$

These expressions coincide with (7.7*b*) and (7.13) if in the first we introduce the change of parameters

$$J = \frac{2f}{ABC} \qquad c = \tfrac{1}{2}ABCc_1 \tag{7.18a}$$

while in the latter we put

$$V_0 = \tfrac{1}{2}c_1 D \qquad \boldsymbol{m} = \boldsymbol{0}. \tag{7.18b}$$

Under the conditions (7.18*b*) the equations of motion in the Euler–Poisson form (7.2) can be written as

$$\dot{\boldsymbol{\omega}}\boldsymbol{I} + \boldsymbol{\omega}\times\boldsymbol{\omega}\boldsymbol{I} = c_1\boldsymbol{\gamma}\times\boldsymbol{\gamma}\boldsymbol{I} \qquad \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega}\times\boldsymbol{\gamma} = \boldsymbol{0}. \tag{7.19}$$

The last equations are well known as describing three physically different integrable problems in rigid body dynamics:

(i) Clebsch's case of motion of a rigid body bounded by a simply connected surface in an infinite ideal incompressible fluid [21] (see [4(*b*)] for this form of the equations of motion).

(ii) The problem of motion of a body about a fixed point under the action of the approximate field of a sufficiently distant Newtonain centre of attraction [22].

(iii) Brun's problem of motion of a body about a fixed point under the assumption that every element of the body is attracted to a fixed plane passing through the fixed point by a force that is proportional to the distance to that plane [23].

The general explicit solution of Clebsch's equivalent of equations (7.19) has been obtained by Kötter [24] in terms of theta functions of two arguments. The special version $f=0$ was solved earlier by Weber [25].

There is still another interpretation that belongs to particle dynamics. In fact, if we perform the transformation (6.29) we can immediately recognize the variables u, v as the ordinary elliptic coordinates on the ellipsoid $Ax_1^2 + Bx_2^2 + Cx_3^2 = 1$. The new system is just a particle that moves with Jacobi's constant $-c$ under the action of certain potential and Lorentz' forces.

7.2.2. The case of Steklov and Rubanovsky. The second of the known integrable cases in the dynamics of a triaxial body is the case found in its simplest version by Steklov [26] for the simply connected body in a liquid. It was generalized by Kharlamov [27] and Rubanovsky [28] to the case of a multiconnected (perforated) body. The equations of motion can be written in the form (7.2) in which

$$\boldsymbol{M} = \boldsymbol{k} + \nu\boldsymbol{\gamma}\boldsymbol{I}^{-1} \qquad V_0 = 0 \tag{7.20}$$

where $\boldsymbol{k}$ is a constant vector (the gyrostatic moment) and ν is a constant scalar. From (7.2*c*) we get

$$\boldsymbol{m} = \boldsymbol{k} - \frac{\nu}{2}\boldsymbol{\gamma}\,\mathrm{diag}\left(\frac{1}{B}+\frac{1}{C}, \frac{1}{C}+\frac{1}{A}, \frac{1}{B}+\frac{1}{A}\right) \tag{7.21}$$

and hence we obtain for the reduced system

$$V=\frac{1}{2ABC}\left[f\sqrt{\lambda\mu}-k_1\left(\frac{BC(A-\lambda)(A-\mu)}{(A-B)(A-C)}\right)^{1/2}-k_2\left(\frac{AC(\lambda-B)(B-\mu)}{(A-B)(B-C)}\right)^{1/2}\right.$$
$$\left.-k_3\left(\frac{AB(\lambda-C)(\mu-C)}{(B-C)(A-C)}\right)^{1/2}+\frac{\nu}{2}\frac{\lambda+\mu}{\sqrt{\lambda\mu}}\right]^2 \tag{7.22a}$$

$$\Omega=\frac{(\lambda-\mu)\sqrt{\lambda\mu}}{4ABC}\left\{f[A+B+C-2(\lambda+\mu)]-k_1[B+C-2(\lambda+\mu)]\gamma_1\right.$$
$$-k_2[A+C-2(\lambda+\mu)]\gamma_2-k_3[A+B-2(\lambda+\mu)]\gamma_3$$
$$\left.+\frac{\nu}{2\lambda\mu}[(A+B+C)(\lambda+\mu)-2(\lambda^2+\lambda\mu+\mu^2)]\right\}. \tag{7.22b}$$

These expressions can be reconstructed from (6.26b) and (6.26c) by taking $\alpha_4=0$ as the double root of the polynomial F and setting

$$a_5=-1/ABC \qquad \alpha_1=A \qquad \alpha_2=B \qquad \alpha_3=C \qquad J_4=f/2 \qquad J_5=\nu/2$$
$$J_1=-\frac{k_1}{2}\left(\frac{BC}{(A-B)(A-C)}\right)^{1/2} \qquad J_2=\frac{k_2}{2\mathrm{i}}\left(\frac{AC}{(A-B)(B-C)}\right)^{1/2}$$
$$J_3=-\frac{k_3}{2}\left(\frac{AB}{(A-C)(B-C)}\right)^{1/2}.$$

Thus, the system of section 6.2.3 is a generalization of Steklov-Rubanovsky's case. The same case can equally be interpreted as a case of motion of an electrically charged body-gyrostat in a uniform magnetic field [4a, b]. If in the above formulae we put $\nu=0$ we obtain the case of an ordinary free gyrostat known after Joukovsky [29].

Explicit solution of Steklov's case ($\boldsymbol{k}=0$, $\nu\neq 0$) was given by Kötter in terms of theta functions of two variables [30]. Joukovsky's case ($\boldsymbol{k}\neq 0$, $\nu=0$) was solved in Weierstrass' functions by Volterra [31] and in a simpler way in terms of Jacobi's elliptic functions by Wittenburg [32]. The case $\nu\boldsymbol{k}\neq 0$ was not, as far as we know, considered.

7.2.3. A new integrable case. The third case is obtained from that of section 6.3 by setting the coefficients in (6.34) so that $F(\lambda)$ is identical with (7.16). We then have from (6.36)

$$V=N(\lambda+\mu)+\frac{J^2}{2ABC}(\lambda+\mu)^2 \tag{7.23a}$$

$$\Omega=\frac{J(\lambda-\mu)}{2ABC}[AB+BC+CA-2(A+B+C)(\lambda+\mu)+3\lambda^2+4\lambda\mu+3\mu^2]. \tag{7.23b}$$

Comparing with (7.7b) and (7.13) we realize that the present case is conditional. It admits the quadratic integral only on the zero level of the parameter f. It is also possible to show that in the equivalent Euler-Poisson form we must have

$$V_0=Nv \tag{7.24a}$$

$$\boldsymbol{M}=\frac{J}{\sqrt{ABCD}}\left\{\left[3v^2-2(A+B+C)v-\frac{2ABC}{D}\right.\right.$$
$$\left.\left.+2(AB+BC+CA)\right]\boldsymbol{\gamma}-v\boldsymbol{\gamma}\boldsymbol{I}-2ABC\boldsymbol{\gamma}\boldsymbol{I}^{-1}\right\} \tag{7.24b}$$

where

$$v=\frac{A(B+C)\gamma_1^2+B(C+A)\gamma_2^2+C(A+B)\gamma_3^2}{A\gamma_1^2+B\gamma_2^2+C\gamma_3^2}.$$

The problem admits a quadratic integral under the condition that the linear integral of areas

$$\omega I\cdot\gamma+\frac{J}{\sqrt{ABCD}}[A(B+C)\gamma_1^2+B(C+A)\gamma_2^2+C(A+B)\gamma_3^2] \tag{7.25}$$

is confined to its zero level. When $J=0$, the present system degenerates into a Liouville one and a separation solution is possible as in section 3.

Thus we have deduced as special cases of the results of section 6 the two known general integrable cases of rigid body dynamics and found a new conditional one.

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